Can an observer really catch up with light? *

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Abstract

Given a null geodesic $\gamma_0(\lambda)$ with a point r in (p, q) conjugate to p along $\gamma_0(\lambda)$, there will be a variation of $\gamma_0(\lambda)$ which will give a time-like curve from p to q. This is a well-known theory proved in the famous book[3]. In the paper we prove that the time-like curves coming from the above-mentioned variation have a proper acceleration which approaches infinity as the time-like curve approaches the null geodesic. This means no observer can be infinitesimally near the light and begin at the same point with the light and finally catch the light. Only separated from the light path finitely, does the observer can begin at the same point with the light and finally catch the light.

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It is well-known that an observer in "hyperbolic" motion in Minkowski space-time has a constant proper acceleration (the magnitude of the 4-acceleration). The equations of the world line of one of such observers can be expressed concisely as $x = x_0$, $y = y_0$, $z = z_0$, (and the proper acceleration $A = x_0^{-1}$.) where $t,\ x,\ y,\ z$ are the Rindler coordinates, and the line element in these coordinates of the Minkowski metric reads

$$ds^2 = -x^2 dt^2 + dx^2 + dy^2 + dz^2. \tag{1}$$

$$(-\infty < t < +\infty,\ x > 0,\ -\infty < y < +\infty,\ -\infty < z < +\infty.)$$

Consider a one-parameter family of hyperbolic observers (x_0, being the parameter) with the same y_0 and z_0, then the proper acceleration, $A = x_0^{-1}$, approaches infinity as x_0 approaches zero. As a limit case, the curve defined by $x = 0$, $y = y_0$, $z = z_0$ is a null geodesic. Unlike time-like curves, the concept of 4-acceleration of a null geodesic (or even a null curve), to our knowledge, has not been defined. The fact that $A \to \infty$ as $x_0 \to 0$, however, suggests that it seems not unreasonable to define the proper acceleration of a null geodesic (or physically, a photon) in Minkowski space-time to be infinity. This is indeed what Rindler suggested in his book [1]. The main purpose of the previous paper[2] is to generalize this result to curved space-times, namely, to argue that it is not unreasonable to define the proper acceleration of a null geodesic which is future-complete in curved space-time to be infinity. The present paper continues the main point of the paper [2], and primely concerns the proper acceleration of time-like curve coming from the variation of null geodesic with two end points fixed on the null geodesic, and gives the conclusion that the proper acceleration of this type of time-like curve does approaches infinity as the time-like curve approaches the null geodesic. Because the two end points fixed on the null geodesic, the existence of the time-like curves from variation of γ_0 is in question. There are two theorems that concern the existence of the time-like curves:

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theorem1 [3],[4]. Let $\gamma_0(\lambda)$ be a smooth causal curve and let $p,\ q \in \gamma_0(\lambda)$. Then there does not exist a smooth one-parameter family of causal curves $\sigma\,(u,\lambda)$ connecting p and q with $\sigma\,(0,\lambda) = \gamma_0(\lambda)$ and $\gamma_u(\lambda)$ time-like for all $u > 0$ (i.e., $\gamma_0(\lambda)$ cannot be smoothly deformed to a time-like curve) if and only if $\gamma_0(\lambda)$ is null geodesic with no point conjugate to p along $\gamma_0(\lambda)$ between p and q.

theorem2[3],[4].If there is a point r in (p,q) conjugate to p along $\gamma_0(\lambda)$, then there will be a variation of $\gamma_0(\lambda)$ which will give a time-like curve from p to q.

We therefore suppose that $\gamma_0(\lambda)$ is a null geodesic with a point r in (p,q) conjugate to p along $\gamma_0(\lambda)$ and the existence of the time-like curves connecting p,q obtained from variation is ensured. Precisely ,we have the following definition of the variation of $\gamma_0(\lambda)$:

Let (M, g_{ab}) be a 4-dimensional curved space-time and γ_0: $(0,\lambda_q)\to M$ be a null geodesic, which will later be denoted by $\gamma_0(\lambda)$ with λ its affine parameter, and with $p,\ q \in \gamma_0(\lambda)$. We define a variation of γ_0 to be a C^{1-} map [3] σ :$[0,\varepsilon) \times [0,\lambda_q] \to M$ such that

(1)$\sigma\,(0,\lambda) = \gamma_0\,(\lambda)$,

(2)$\sigma\,(u,0) = p,\ \sigma\,(u,\lambda_q) = q$

(3)there is a subdivision $0 = \lambda_1 < \lambda_2 < ... < \lambda_n = \lambda_q$ of $[0,\lambda_q]$ such that σ is C^3 on each $[0,\varepsilon) \times [\lambda_i, \lambda_{i+1}]$.

(4)for each constant $u \in [0,\varepsilon)$ and $u \neq 0$, $\sigma\,(u,\lambda)$ is a time-like curve and is represented by $\gamma_u\,(\lambda)$.

Denote by $\left(\frac{\partial}{\partial\lambda}\right)_u^a \equiv v_u^a$ the tangent vector to the curve $\gamma_u(\lambda)$, then $\left(\frac{\partial}{\partial\lambda}\right)_0^a \equiv v_0^a$ satisfies the null geodesic equation:

$$\left(\frac{\partial}{\partial\lambda}\right)_0^b \nabla_b \left(\frac{\partial}{\partial\lambda}\right)_0^a = 0, \tag{2}$$

where ∇_a is the unique derivative operator associated with g_{ab}, i.e., $\nabla_a g_{bc} = 0$. We rewrite eq.(2)as

$$v_0^b \nabla_b v_0^a = 0 \tag{3}$$

Let $\left(\frac{\partial}{\partial u}\right)^a$ be the tangent vector to the curve $\sigma(u,\lambda)$ with $\lambda = const$, and define the variation vector field Z^a on $\gamma_0(\lambda)$ by

$$Z^a = \left(\frac{\partial}{\partial u}\right)^a |_{u=0}, \tag{4}$$

then it is not difficult to see that the Lie derivative of $\left(\frac{\partial}{\partial u}\right)^a$ with respect to $\left(\frac{\partial}{\partial\lambda}\right)^a$ vanishes [3],i.e.,

$$L_{\frac{\partial}{\partial\lambda}} \left(\frac{\partial}{\partial u}\right)^a = 0. \tag{5}$$

that is

$$v_u^b \nabla_b \left(\frac{\partial}{\partial u}\right)^a = \left(\frac{\partial}{\partial u}\right)^b \nabla_b v_u^a, \tag{6}$$

evaluation of equation (6) on $\gamma_0(\lambda)$ yields

$$v_0^b \nabla_b Z^a = Z^b \nabla_b v_0^a, \tag{7}$$

If we denote $g_{ab} v_u^a v_u^b$ by $-\alpha_u^2$, that is

$$-\alpha_u^2 = g_{ab} v_u^a v_u^b, \tag{8}$$

and decompose $-\alpha_u^2$ into Taylor series

$$-\alpha_u^2 = g_{ab} v_u^a v_u^b = -\alpha_0^2 + \beta_1 u + \frac{1}{2}\beta_2 u^2 + 0(u^3), \tag{9}$$

where

$$\alpha_0^2 = g_{ab} v_0^a v_0^b = 0. \tag{10}$$

In order to get $\gamma_u$to be time-like, it is easy to see $\beta_1 = \frac{\partial(-\alpha^2)}{\partial u}|_{u=0} \leq 0$ and it can be prove $\beta_1 = 0$ (see detail in reference [3]). The following is the reason: with eq(6)

$$\begin{aligned}
\frac{\partial(-\alpha_u^2)}{\partial u} &= (\frac{\partial}{\partial u})^c \nabla_c \left[g_{ab} v_u^a v_u^b\right] = 2 g_{ab} v_u^a (\frac{\partial}{\partial u})^c \nabla_c v_u^b = 2 g_{ab} v_u^a v_u^c \nabla_c (\frac{\partial}{\partial u})^b \\
&= 2 v_u^c \nabla_c \left[g_{ab} v_u^a (\frac{\partial}{\partial u})^b\right] - 2 g_{ab} (\frac{\partial}{\partial u})^b v_u^c \nabla_c v_u^a \\
&= 2 \frac{\partial}{\partial \lambda} \left[g_{ab} v_u^a (\frac{\partial}{\partial u})^b\right] - 2 g_{ab} (\frac{\partial}{\partial u})^b v_u^c \nabla_c v_u^a,
\end{aligned} \tag{11}$$

therefore, with eq.(3), one gets

$$\beta_1 = \frac{\partial(-\alpha_u^2)}{\partial u}|_{u=0} = 2 \frac{\partial}{\partial \lambda} \left[g_{ab} v_0^a Z^b\right] = 2 \frac{dh}{d\lambda} \tag{12}$$

where $h(\lambda) = g_{ab} v_0^a Z^b$ is continuous on $(0, \lambda_q)$ thanks to the continuity of of Z^a . From the property of the variation map $\sigma(u, \lambda)$, that is , $\sigma(u, 0) = p$, $\sigma(u, \lambda_q) = q$, we get

$$Z^a(0) = Z^a(\lambda_q) = 0. \tag{13}$$

This in turn induces $h(0) = h(\lambda_q) = 0$, but $h(0) = h(\lambda_q) = 0$ is impossible if β_1 is less than zero. So, β_1 must be zero to make γ_u time-like.

$\beta_1 = 0$ induces $h = g_{ab} v_0^a Z^b = 0$, that is, the variation vector Z^a is orthogonal to the tangent vector v_0^a of the null geodesic γ_0.

Therefore, one gets

$$-\alpha_u^2 = g_{ab} v_u^a v_u^b = \frac{1}{2} \beta_2 u^2 + 0(u^3), \tag{14}$$

$\beta_2 = \frac{1}{2} \frac{\partial^2(-\alpha_u^2)}{\partial u^2}|_{u=0}$ is followed from eq.(11)(see detail in reference [3])

$$\begin{aligned}
\frac{1}{2} \frac{\partial^2(-\alpha_u^2)}{\partial u^2} &= \frac{\partial^2}{\partial \lambda \partial u} \left[g_{ab} v_u^a (\frac{\partial}{\partial u})^b\right] - (\frac{\partial}{\partial u})^d \nabla_d \left[g_{ab} (\frac{\partial}{\partial u})^b v_u^c \nabla_c v_u^a\right] \\
= \frac{\partial^2}{\partial \lambda \partial u} &\left[g_{ab} v_u^a (\frac{\partial}{\partial u})^b\right] - \left[g_{ab} v_u^c \nabla_c v_u^a (\frac{\partial}{\partial u})^d \nabla_d (\frac{\partial}{\partial u})^b\right] - \left[(\frac{\partial}{\partial u})^a (\frac{\partial}{\partial u})^d \nabla_d \left(v_u^c \nabla_c v_{ua}\right)\right]
\end{aligned} \tag{15}$$

the term $(\frac{\partial}{\partial u})^d \nabla_d (v_u^c \nabla_c v_{ua})$ in the third part of above equation is simplified:

$$\begin{aligned}
(\frac{\partial}{\partial u})^d \nabla_d \left(v_u^c \nabla_c v_{ua}\right) &= \left((\frac{\partial}{\partial u})^d \nabla_d v_u^c\right) \nabla_c v_{ua} + v_u^c (\frac{\partial}{\partial u})^d \nabla_d \nabla_c v_{ua} \\
&= \left((\frac{\partial}{\partial u})^d \nabla_d v_u^c\right) \nabla_c v_{ua} + v_u^c (\frac{\partial}{\partial u})^d \nabla_c \nabla_d v_{ua} + R_{dcae} v_u^c v_u^e (\frac{\partial}{\partial u})^d \\
&= \left((\frac{\partial}{\partial u})^d \nabla_d v_u^c\right) \nabla_c v_{ua} + v_u^c \nabla_c \left[(\frac{\partial}{\partial u})^d \nabla_d v_{ua}\right] \\
&\quad - \left(v_u^c \nabla_c (\frac{\partial}{\partial u})^d\right) \nabla_d v_{ua} + R_{dcae} v_u^c v_u^e (\frac{\partial}{\partial u})^d \\
&= v_u^d \nabla_d \left(v_u^c \nabla_c (\frac{\partial}{\partial u})_a\right) + R_{dcae} v_u^c v_u^e (\frac{\partial}{\partial u})^d,
\end{aligned}$$

that is

$$(\frac{\partial}{\partial u})^d \nabla_d \tilde{A}_{ua} = (\frac{\partial}{\partial u})^d \nabla_d \left(v_u^c \nabla_c v_{ua}\right) = v_u^d \nabla_d \left(v_u^c \nabla_c (\frac{\partial}{\partial u})_a\right) + R_{dcae} v_u^c v_u^e (\frac{\partial}{\partial u})^d. \tag{16}$$

where the relation $\nabla_c \nabla_d v_{ua} - \nabla_d \nabla_c v_{ua} = -R_{dcae} v_u^e$ in second step and eq.(6)in the fourth step have been used, and $\tilde{A}_{ua}$ is defined by $\tilde{A}_{ua} = v_u^c \nabla_c v_{ua}$ and different from the proper acceleration of the time-like curve $\gamma_u(\lambda)$ (see following for the definition of the proper acceleration of $\gamma_u(\lambda)$). Therefore, one gets

$$\begin{aligned}
\frac{1}{2} \frac{\partial^2(-\alpha_u^2)}{\partial u^2} &= \frac{\partial^2}{\partial \lambda \partial u} \left[g_{ab} v_u^a (\frac{\partial}{\partial u})^b\right] - \left[g_{ab} \left(v_u^c \nabla_c v_u^a\right) (\frac{\partial}{\partial u})^d \nabla_d (\frac{\partial}{\partial u})^b\right] \\
&- \left[(\frac{\partial}{\partial u})^a v_u^d \nabla_d \left(v_u^c \nabla_c (\frac{\partial}{\partial u})_a\right) + R_{dcae} v_u^c v_u^e (\frac{\partial}{\partial u})^a (\frac{\partial}{\partial u})^d\right],
\end{aligned} \tag{17}$$

so,

$$\beta_2 = \left[\frac{1}{2}\frac{\partial^2(-\alpha_u^2)}{\partial u^2}\right]_{u=0} = \left[\frac{\partial^2}{\partial\lambda\partial u}\left(g_{ab}v_u^a(\frac{\partial}{\partial u})^b\right)\right]_{u=0} - \left[Z^a v_0^d \nabla_d\left(v_0^c \nabla_c Z_a\right) + R_{dcae} v_0^c v_0^e Z^d Z^a\right], \tag{18}$$

where eq. (3) has been used. for simplicity, one has

$$\begin{aligned}
\tilde{\beta}_2 &= \left[\frac{\partial^2}{\partial\lambda\partial u}(g_{ab}v_u^a(\frac{\partial}{\partial u})^b)\right]_{u=0} = \frac{\partial}{\partial\lambda}\left[(\frac{\partial}{\partial u})^c\nabla_c(g_{ab}v_u^a(\frac{\partial}{\partial u})^b)\right]_{u=0} \\
&= \frac{\partial}{\partial\lambda}\left[g_{ab}(\frac{\partial}{\partial u})^b(\frac{\partial}{\partial u})^c\nabla_c v_u^a\right]_{u=0} + \frac{\partial}{\partial\lambda}\left[g_{ab}v_u^a(\frac{\partial}{\partial u})^c\nabla_c(\frac{\partial}{\partial u})^b\right]_{u=0} \\
&= \frac{\partial}{\partial\lambda}\left[g_{ab}(\frac{\partial}{\partial u})^b v_u^c\nabla_c(\frac{\partial}{\partial u})^a\right]_{u=0} = \frac{1}{2}\frac{\partial^2}{\partial\lambda^2}\left[g_{ab}(\frac{\partial}{\partial u})^b(\frac{\partial}{\partial u})^a\right]_{u=0} \\
&= \frac{1}{2}\frac{d^2}{d\lambda^2}\left(g_{ab}Z^aZ^b\right),
\end{aligned} \tag{19}$$

where the relation $\left[(\frac{\partial}{\partial u})^c\nabla_c(\frac{\partial}{\partial u})^b\right]_{u=0} = 0$ has been used, which comes from the relation of the variation map $\sigma(u,\lambda)$ and the variation vector Z^a: $\sigma(u,\lambda) = exp_r(uZ^a)$, $r = \gamma_0(\lambda)$ with $u = const$ (see page 107 of the reference [3]).

$$\beta_2 = \frac{1}{2}\frac{d^2}{d\lambda^2}\left(g_{ab}Z^aZ^b\right) - \left[Z^a v_0^d \nabla_d\left(v_0^c \nabla_c Z_a\right) + R_{dcae} v_0^c v_0^e Z^d Z^a\right]. \tag{20}$$

The parameter λ of the timelike curve, $\gamma_u(\lambda)$, defined above is not, in general, the proper time of the curve. If one re-parameterizes the curve $\gamma_u(\lambda)$ by its proper time τ, i.e., the parameter satisfying

$$g_{ab}\left(\frac{\partial}{\partial\tau}\right)_u^a\left(\frac{\partial}{\partial\tau}\right)_u^b = -1, \tag{21}$$

then

$$\left(\frac{\partial}{\partial\tau}\right)^a = \left(\frac{d\lambda}{d\tau}\right)\left(\frac{\partial}{\partial\lambda}\right)_u^a = \left(\frac{d\lambda}{d\tau}\right)v_u^a. \tag{22}$$

With eq.(8), one has

$$\left(\frac{d\lambda}{d\tau}\right)^2 = \frac{1}{\alpha_u^2}. \tag{23}$$

The 4-acceleration of the time-like curve γ_u which is defined as

$$A^a = \left(\frac{\partial}{\partial\tau}\right)^b\nabla_b\left(\frac{\partial}{\partial\tau}\right)^a = \frac{d\lambda}{d\tau}v_u^b\nabla_b\left(\frac{d\lambda}{d\tau}v_u^a\right) = \left(\frac{1}{\alpha_u^2}\right)\widetilde{A}_u^a - \frac{1}{2\alpha_u^4}v_u^a v_u^b\nabla_b\alpha_u^2,, \tag{24}$$

or, one writes the above equation as

$$\widetilde{A}_u^a = \alpha_u^2 A^a + \frac{1}{2\alpha_u^2}v_u^a v_u^b\nabla_b\alpha_u^2,. \tag{25}$$

with eq.(25), then

$$(\frac{\partial}{\partial u})^a(\frac{\partial}{\partial u})^d\nabla_d(\alpha_u^2 A_a) = (\frac{\partial}{\partial u})^a(\frac{\partial}{\partial u})^d\nabla_d\tilde{A}_{ua} - (\frac{\partial}{\partial u})^a(\frac{\partial}{\partial u})^d\nabla_d\left(\frac{1}{2\alpha_u^2}v_{ua}v_u^b\nabla_b\alpha_u^2\right), \tag{26}$$

first, the calculation of $b_1 \equiv (\frac{\partial}{\partial u})^a(\frac{\partial}{\partial u})^d\nabla_d\left(\frac{1}{2\alpha_u^2}v_{ua}v_u^b\nabla_b\alpha_u^2\right)$ is

$$\begin{aligned}
b_1 &= (\frac{\partial}{\partial u})^a v_{ua}(\frac{\partial}{\partial u})^d\nabla_d\left(\frac{1}{2\alpha_u^2}v_u^b\nabla_b\alpha_u^2\right) + \frac{1}{2\alpha_u^2}(v_u^b\nabla_b\alpha_u^2)(\frac{\partial}{\partial u})^a(\frac{\partial}{\partial u})^d\nabla_d v_{ua} \\
&= (\frac{\partial}{\partial u})^a v_{ua}(\frac{\partial}{\partial u})^d\nabla_d\left(\frac{1}{2\alpha_u^2}v_u^b\nabla_b\alpha_u^2\right) + \frac{1}{2\alpha_u^2}(v_u^b\nabla_b\alpha_u^2)(\frac{\partial}{\partial u})^a v_u^d\nabla_d(\frac{\partial}{\partial u})_a \\
&= (\frac{\partial}{\partial u})^a v_{ua}(\frac{\partial}{\partial u})^d\nabla_d\left(\frac{1}{2\alpha_u^2}v_u^b\nabla_b\alpha_u^2\right) + \frac{1}{4\alpha_u^2}(v_u^b\nabla_b\alpha_u^2)v_u^d\nabla_d[(\frac{\partial}{\partial u})_a(\frac{\partial}{\partial u})^a].
\end{aligned} \tag{27}$$

with eq. (13), therefore,

$$\lim_{u\to 0} b_1 = \lim_{u\to 0}(\frac{1}{4\alpha_u^2} v_u^b \nabla_b \alpha_u^2) v_0^d \nabla_d (Z^a Z_a) = \lim_{u\to 0}(\frac{1}{4\alpha_u^2} v_u^b \nabla_b \alpha_u^2) \frac{d}{d\lambda}(Z^a Z_a) \tag{28}$$

second, with eqs.(16),(18), (20), calculate $b_2 \equiv (\frac{\partial}{\partial u})^a (\frac{\partial}{\partial u})^d \nabla_d \tilde{A}_{ua}$

$$\begin{aligned} b_2 &= (\frac{\partial}{\partial u})^a [v_u^d \nabla_d \left(v_u^c \nabla_c (\frac{\partial}{\partial u})_a \right) + R_{dcae} v_u^c v_u^e (\frac{\partial}{\partial u})^d] \\ \lim_{u\to 0} b_2 &= \left[Z^a v_0^d \nabla_d \left(v_0^c \nabla_c Z_a \right) + R_{dcae} v_0^c v_0^e Z^d Z^a \right] \\ &= \frac{1}{2}\frac{d^2}{d\lambda^2}\left(g_{ab} Z^a Z^b \right) - \beta_2. \end{aligned} \tag{29}$$

thirdly, $b \equiv (\frac{\partial}{\partial u})^a (\frac{\partial}{\partial u})^d \nabla_d (\alpha_u^2 A_a)$ is

$$\begin{aligned} b &= (\frac{\partial}{\partial u})^d \nabla_d (\alpha_u^2 A_a (\frac{\partial}{\partial u})^a) - (\alpha_u^2 A_a)(\frac{\partial}{\partial u})^d \nabla_d (\frac{\partial}{\partial u})^a \\ \lim_{u\to 0} b &= \lim_{u\to 0}(\frac{\partial}{\partial u})^d \nabla_d (\alpha_u^2 A_a (\frac{\partial}{\partial u})^a), \end{aligned} \tag{30}$$

where one uses the relation $\lim_{u\to 0}[(\frac{\partial}{\partial u})^d \nabla_d (\frac{\partial}{\partial u})^a] = 0$. The following shows that $\lim_{u\to 0} b = \lim_{u\to 0} b_2 - \lim_{u\to 0} b_1 \neq 0$. With eqs.(14), (28),(29), then

$$\lim_{u\to 0} b = \lim_{u\to 0} b_2 - \lim_{u\to 0} b_1 = \frac{1}{2}\frac{d^2}{d\lambda^2}\left(g_{ab} Z^a Z^b \right) - \beta_2 - \lim_{u\to 0}(\frac{1}{4\alpha_u^2} v_u^b \nabla_b \alpha_u^2)\frac{d}{d\lambda}(Z^a Z_a) \tag{31}$$

$$= \frac{1}{2}\frac{d^2}{d\lambda^2}\left(g_{ab} Z^a Z^b \right) - \frac{1}{4\beta_2}\frac{d\beta_2}{d\lambda}\frac{d}{d\lambda}(Z^a Z_a) - \beta_2 \tag{32}$$

$$= \frac{1}{2}\frac{(-\beta_2)}{(-\beta_2)^{\frac{1}{2}}}\frac{d}{d\lambda}\left[\frac{\frac{d}{d\lambda}(Z^a Z_a)}{(-\beta_2)^{\frac{1}{2}}} \right] - \beta_2. \tag{33}$$

If $\lim_{u\to 0} b = 0$, from eq.(33), one obtains $\frac{1}{2}\frac{d}{d\lambda}\left[\frac{\frac{d}{d\lambda}(Z^a Z_a)}{(-\beta_2)^{\frac{1}{2}}} \right] = -(-\beta_2)^{\frac{1}{2}}$. Because Z^a is a continuous, piecewise C^2 vector field along $\gamma_0(\lambda)$ vanishing at end points p and q, $\frac{d(Z^a Z_a)}{d\lambda} = 0$ must be satisfied at end points p and q, and in the first interval $[0, \lambda_2]$,Z^a is C^2, then $\forall \lambda \in [0, \lambda_2]$,

$$\int_0^\lambda -2(-\beta_2)^{\frac{1}{2}} d\lambda = \left[\frac{\frac{d}{d\lambda}(Z^a Z_a)}{(-\beta_2)^{\frac{1}{2}}} \right]_\lambda, \tag{34}$$

$$\left[\frac{d(Z^a Z_a)}{d\lambda} \right]_\lambda = -2(-\beta_2)^{\frac{1}{2}}(\lambda)\int_0^\lambda (-\beta_2)^{\frac{1}{2}} d\lambda \tag{35}$$

because $\gamma_u(\lambda)$ is time-like curve, $(-\beta_2)^{\frac{1}{2}} > 0$ is satisfied everywhere, this in turn ensures $\frac{d}{d\lambda}(Z^a Z_a) < 0$ everywhere in the interval $[0, \lambda_2]$. This is impossible, as $(Z^a Z_a)\mid_{\lambda=0}= 0$, in the neighborhood of the initial point p, Z^a varies from zero to $Z^a \neq 0$, and Z^a is space-like, so, $(Z^a Z_a)$ increases with λ in the neighborhood of the initial point p , that is , $\frac{d}{d\lambda}(Z^a Z_a)$ must be larger than zero in the neighborhood of the initial point p. Therefore, $\lim_{u\to 0} b = \lim_{u\to 0} b_2 - \lim_{u\to 0} b_1 \neq 0$ is guaranteed. From eq.(30), one gets $\lim_{u\to 0}(\frac{\partial}{\partial u})^d \nabla_d (\alpha_u^2 A_a (\frac{\partial}{\partial u})^a)$ is not equal to zero, as $\alpha_u^2 = \frac{1}{2}\beta_2 u^2$ approaches zero when $u \to 0$, these insures that $A^a Z_a$ approaches infinity as $u \to 0$, which induce $A^a \to \infty$ due to the finiteness of the variation vector Z^a.

In conclusion, When a null geodesic γ_0 connecting p, q with a point $r \in (p, q)$ conjugate to p, then the proper acceleration of the time-like curve from p to q produced from the variation of the γ_0 approaches infinity as $u \to 0$, this means no observer can be infinitesimally near the light and begin at the same point with the light and finally catch the light. Only separated from the light path finitely, does the observer can begin at the same point with the light and finally catch the light.

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